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A NOTE ON THE OPTIMALITY OF A SWITCH-OVER POLICY FOR THE  
M/G/1 QUEUE WITH VARIABLE SERVICE RATE

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A note on the optimality of a switch-over policy for the M/G/1 queue  
with variable service rate

by

H.C. Tijms

#### ABSTRACT

This note considers the M/G/1 queue in which a finite number of service types are available. There is a linear holding cost rate, and a fixed reward for each customer served. The purpose of this note is to point out that under the assumption of stochastically ordered service times there is an average cost optimal stationary policy having the property that the service type used is a non-decreasing function of the queue size.

KEY WORDS & PHRASES: *M/G/1 queue with variable service rate, switch-over policy, average cost optimal.*



## 1. INTRODUCTION

Consider a single-server station where customers arrive in accordance with a Poisson process with rate  $\lambda$ . For each new service to be started the server must choose one of a finite number of different service types  $k=1, \dots, M$ . For service type  $k$  the service time is a positive random variable  $S_k$  with probability distribution function  $F_k(t)$ . It is assumed that  $S_k$  is stochastically smaller than  $S_j$  for all  $k$  and  $j$  with  $k > j$ , that is,  $F_k(t) \geq F_j(t)$  for  $t \geq 0$  when  $k > j$ , so type  $k$  is "faster" than type  $j$  for  $k > j$ . Further we assume  $ES_k^2 < \infty$  for all  $k$ ,  $\lambda ES_M < 1$  and  $ES_M^3 < \infty$ . The following costs are considered. There is a holding cost of  $h > 0$  per customer per unit time, a service cost at rate  $r_k$  when the server is busy and uses service type  $k$ , a service cost at rate  $r_0$  when the server is idle, and a fixed reward  $R_k$  for each customer served by using service type  $k$ .

Define the state of the system as the number of customers present. The system is only observed at the epochs where a new service must be started and the epochs where the server becomes idle. When the system is observed in state  $i \geq 1$ , then one of the actions  $k=1, \dots, M$  must be chosen where the choice of action  $k$  means that service type  $k$  is used for the new service to be started. For notational purposes, we say that action 0 is chosen when state 0 is observed. Let  $C(i, k)$  be the expected cost incurred until the next review when in state  $i$  action  $k$  is chosen. Then,  $C(i, k) = hiES_k + h\lambda ES_k^2/2 + r_k ES_k - R_k$  for all  $i \geq 1$  and  $1 \leq k \leq M$ , and  $C(0, 0) = r_0/\lambda$ . Since we will consider the average cost criterion, it is no restriction to assume that immediate costs  $C(i, k)$  are incurred when action  $k$  is taken in state  $i$ . A policy  $\pi$  is any rule for choosing actions, where a policy  $f$  is said to be stationary if it chooses a single action  $f(i)$  whenever the system is in

state  $i$ . A stationary policy  $f$  is called a switch-over policy when  $f(i)$  is non-decreasing in  $i \geq 1$ .

Let  $V(\pi, i, t)$  be the total expected cost incurred in  $[0, t)$  when policy  $\pi$  is used and the initial state is  $i$ , and, for any  $i$  and  $\pi$ , let

$$\bar{V}(\pi, i) = \limsup_{t \rightarrow \infty} t^{-1} V(\pi, i, t) \text{ and } V_{\alpha}(\pi, i) = \int_0^{\infty} e^{-\alpha t} dV(\pi, i, t)$$

for  $\alpha > 0$ , so, for initial state  $i$  and policy  $\pi$ ,  $\bar{V}(\pi, i)$  is the long-run average cost and  $V_{\alpha}(\pi, i)$  is the expected total discounted cost when the discount factor is  $\alpha$ . A policy  $\pi^*$  is called average cost optimal when  $\bar{V}(\pi^*, i) \leq \bar{V}(\pi, i)$  for all  $i$  and  $\pi$ , and a policy  $\pi^*$  is called  $\alpha$ -optimal when  $V_{\alpha}(\pi^*, i) \leq V_{\alpha}(\pi, i)$  for all  $i$  and  $\pi$ . Let  $V_{\alpha}(i) = \inf_{\pi} V_{\alpha}(\pi, i)$ ,  $i \geq 0$ .

The existence of an average cost optimal switch-over policy was shown in CRABILL[1] and in LIPPMAN[4,5] for the case where the service times are exponential and the service rate can also be chosen at arrival epochs. SCHASSBERGER[8] considered the case of stochastically ordered service times and, assuming a finite waiting room and no holding cost, he proved that there is an average cost optimal switch-over policy. His proof, however, fails for the model of this paper. The purpose of this note is to point out that using recent work of LIPPMAN[4,6] the average cost optimality of a switch-over policy can be readily shown.

## 2. PROOF

We first give some preliminaries. The notation  $X \leq Y$  means that the random variable  $X$  is stochastically smaller than the random variable  $Y$ . We have (see[9]).

LEMMA 1. Let  $X < Y$ . Then, for any non-decreasing function  $f$ ,  $Ef(X) \leq Ef(Y)$  provided the expectations exist.

Let  $A_k$  be distributed as the number of arrivals during a service time  $S_k$ . Since  $P\{A_k > n\} = \int_0^\infty P\{A_k > n \mid S_k = t\} dF_k(t)$ , lemma 1 implies

LEMMA 2.  $A_k < A_j$  for all  $k$  and  $j$  with  $k > j$ .

Let  $T$  be the epoch of the first return of the system to state 0, and let  $Z(t)$  be the total costs incurred during  $[0, t)$ ,  $t \geq 0$ . Denote by  $E_{\pi, i}$  the expectation when policy  $\pi$  is used and the initial state is  $i$ .

LEMMA 3. Let  $f$  be a stationary policy such that  $f(i) = M$  for all  $i$  sufficiently large. Then,  $E_{i, f}(T) \leq \alpha_1 i + \alpha_2$  and  $E_{i, f}(|Z(T)|) \leq \beta_1 i^2 + \beta_2 i + \beta_3$  for all  $i \geq 0$  where the  $\alpha_j$  and  $\beta_k$  are constants.

PROOF. Consider the M/G/1 queue in which the traffic intensity is less than 1 and the service time has a finite second moment. Suppose that at epoch 0 a service starts when  $s \geq 1$  customers are present. From queueing theory it is wellknown that the expectation of the first epoch at which the system becomes empty is a linear function of  $s$  and that the expected total time spent by the customers in the system up to that epoch is a quadratic function of  $s$ . Since  $\lambda ES_M < 1$  and  $ES_M^2 < \infty$ , the lemma now follows easily.  $\square$

To prove that there is an average cost optimal switch-over policy, we first consider the discounted model. For the semi-Markov decision model with unbounded costs HARRISON[2,3] and LIPPMAN[4,6] have given conditions under which for each  $\alpha > 0$  an  $\alpha$ -optimal stationary policy exists and the optimality equation applies. It is straightforward to verify that for this

problem both the conditions in [2] and those in [6] hold. This implies that for any  $\alpha > 0$ ,

$$(1) \quad V_{\alpha}(i) = \min_{1 \leq k \leq M} \{C(i,k) + \int_0^{\infty} e^{-\alpha t} V_{\alpha}(i-1+k) e^{-\lambda t} \frac{(\lambda t)^k}{k!} dF_k(t)\}; \quad i \geq 1,$$

where  $V_{\alpha}(0) = r_0/\lambda + \int_0^{\infty} e^{-\alpha t} V_{\alpha}(1) \lambda e^{-\lambda t} dt$ . Also, for any  $\alpha > 0$ , let  $f_{\alpha}$  be a stationary policy such that  $f_{\alpha}(i)$  minimizes the right side of (1) for all  $i$ , then  $f_{\alpha}$  is  $\alpha$ -optimal. Using the lemmas 1 and 2 and making a minor modification of the first part of the proof of Theorem 6 in [4], we get that there is an  $\alpha^* > 0$  and a bound  $B < \infty$  such that  $f_{\alpha}(i) = M$  for all  $0 < \alpha < \alpha^*$  and  $i > B$ . This implies

LEMMA 4. *There is a stationary policy  $f^*$  with  $f^*(i) = M$  for all  $i > B$  and a sequence  $\{\alpha_k\}$  with  $\alpha_k \rightarrow 0$  as  $k \rightarrow \infty$  such that  $f_{\alpha_k} = f^*$  for all  $k$ .*

The next theorem can be readily obtained from a close examination of the analysis of the average cost criterion in [4,6]. However, since this analysis is rather complicated by its generality and needs some minor modifications, it might be helpful to outline a simple proof that suffices for the present problem.

THEOREM 1. *The policy  $f^*$  is average cost optimal, and  $\bar{V}(f^*, i) = g$  for all  $i$  for some constant  $g$ . There is a function  $h$  with  $h(0) = 0$  and*

$$(2) \quad |h(i)| \leq \alpha i^2 + \beta i + \gamma \quad \text{for } i \geq 0,$$

for some constants  $\alpha$ ,  $\beta$  and  $\gamma$ , such that  $h(0) = r_0/\lambda - g/\lambda + h(1)$  and

$$(3) \quad h(i) = \min_{1 \leq k \leq M} \{C(i,k) - gES_k + \sum_{j=0}^{\infty} h(i-1+j)p_j^{(k)}\} \quad \text{for } i \geq 1,$$



where  $p_j^{(k)} = P\{A_k = j\} = \int_0^\infty e^{-\lambda t} \{(\lambda t)^j / j!\} dF_k(t)$ . Moreover,  $f^*(i)$  minimizes the right side of (3) for all  $i$ .

PROOF Let  $g = E_{0,f^*}(Z(T))/E_{0,f^*}(T)$ . Then, by Lemma 3 and Theorem 3.16 in ROSS[7], we have that  $t^{-1}V(f^*, i, t)$  has the finite limit  $g$  as  $t \rightarrow \infty$  for all  $i$ . Now, from Lemma 4 and a standard Tauberian theorem (see pp. 181-182 in [10]) it follows that, for all  $i$  and  $\pi$ ,

$$\begin{aligned} \bar{V}(\pi, i) &= \limsup_{t \rightarrow \infty} t^{-1}V(\pi, i, t) \geq \limsup_{\alpha \rightarrow 0} \alpha V_\alpha(\pi, i) \geq \\ &\geq \limsup_{k \rightarrow \infty} \alpha_k V_{\alpha_k}(i) = \lim_{k \rightarrow \infty} \alpha_k V_{\alpha_k}(f^*, i) = \lim_{t \rightarrow \infty} t^{-1}V(f^*, i, t). \end{aligned}$$

This proves the first part of the theorem. As a byproduct we find

$$(4) \quad \lim_{k \rightarrow \infty} \alpha_k V_{\alpha_k}(i) = g \quad \text{for all } i \geq 0$$

Following the proof of Theorem 4 in [4] (cf. also p. 148 in [7]) and using (4), we find that, for some constant  $\delta$ ,

$$(5) \quad \delta E_{i,f^*}(T) \leq V_{\alpha_k}(i) - V_{\alpha_k}(0) \leq E_{i,f^*}(Z(T)) \quad \text{for all } k \text{ and all } i,$$

so, for each  $i$ ,  $\{V_{\alpha_k}(i) - V_{\alpha_k}(0)\}$  is a bounded sequence. Now, by Cauchy's diagonalization method, there is a subsequence  $\{\alpha'_k\}$  of  $\{\alpha_k\}$  and a function  $h$  such that

$$(6) \quad h(i) = \lim_{k \rightarrow \infty} \{V_{\alpha'_k}(i) - V_{\alpha'_k}(0)\} \quad \text{for all } i \geq 0.$$

By (5), (6) and Lemma 3 we have that  $h$  satisfies (2). Next we observe that, by (2) and  $ES_k^2 < \infty$ ,

$$(7) \quad \sum_{j=0}^{\infty} |h(i-1+j)| p_j^{(k)} < \infty \quad \text{for all } i \geq 1 \text{ and } 1 \leq k \leq M.$$

Finally, subtracting  $V_\alpha(0)$  from both sides of (1) with  $\alpha = \alpha'_k$ , letting  $k \rightarrow \infty$ , and using (4), (6), (7) and the construction of  $f^*$ , we find the other assertions of the theorem (we note that (7) is needed for applying the bounded convergence theorem).  $\square$

A repetition of the second part of the proof of Theorem 6 in [4] shows

LEMMA 5.  $h(i+1) - h(i) \geq h(i) - h(i-1)$  for all  $i \geq 1$ , i.e. the function  $h$  is convex.

Denote by  $h(i, k)$  the expression between brackets in (3). Then

LEMMA 6. For all  $k_1, k_2$  with  $k_1 > k_2$ ,  $h(i, k_1) - h(i, k_2)$  is non-increasing in  $i \geq 1$ .

PROOF. Fix  $k_1, k_2$  with  $k_1 > k_2$  and fix  $i \geq 1$ . Using (7), we have

$$\begin{aligned} h(i+1, k_1) - h(i+1, k_2) - \{h(i, k_1) - h(i, k_2)\} &= hES_{k_1} - hES_{k_2} + \\ &+ \sum_{j=0}^{\infty} \{h(i+j) - h(i-1+j)\} p_j^{(k_1)} - \sum_{j=0}^{\infty} \{h(i+j) - h(i-1+j)\} p_j^{(k_2)}. \end{aligned}$$

By lemma 5,  $h(i+j) - h(i-1+j)$  is non-decreasing in  $j \geq 0$ . Now, the lemma follows from the Lemmas 1 and 2 and the fact that  $ES_{k_1} < ES_{k_2}$ .  $\square$

We are now in a position to state our main result.

THEOREM 2. For any  $i \geq 1$ , let  $f_0(i)$  be the largest value of  $k$  for which the right side of (3) is minimal. Then,  $f_0$  is an average cost optimal switch-over policy which uses service type  $M$  for all  $i$  sufficiently large.

PROOF. It easily follows from Lemma 6 that  $f_0(i+1) \geq f_0(i)$  for all  $i \geq 1$ , and, by Lemma 4 and Theorem 1,  $f_0(i) = M$  for all  $i$  sufficiently large.

Since  $g$  is the minimal average cost, the switch-over policy  $f_0$  is optimal when  $\bar{V}(f_0, i) = g$  for all  $i$ . To prove this, we first observe that Lemma 3 and the proof of Theorem 7.5 in [7] imply that, for all  $i$ ,

$$(8) \quad \bar{V}(f_0, i) = \bar{V}(f_0, 0) = \lim_{n \rightarrow \infty} E_{0, f_0} \left( \sum_{j=1}^n Z_j \right) / E_{0, f_0} \left( \sum_{j=1}^n \tau_j \right),$$

where  $Z_k$  denotes the cost incurred at the  $(k-1)$ th review and  $\tau_k$  denotes the time between the  $(k-1)$ th and the  $k$ th review. Let  $X_k$  be the state at the  $k$ th review. Since  $\lambda ES_M < 1$  and  $ES_M^3 < \infty$ , it follows from queueing theory that  $E_{0, f_0}(X_k^j)$  has a finite limit as  $k \rightarrow \infty$  for  $j = 1, 2$ . Hence, by (2),  $k^{-1} E_{0, f_0}(h(X_k))$  goes to zero as  $k \rightarrow \infty$ . Now, the proof of Theorem 7.6 in [7] implies that the right side of (8) equals  $g$  (cf. p. 727 in [4]). This completes the proof.  $\square$

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